4.1 Inverse Functions

**Definition:** A function is one to one if and only if every domain element maps to a unique (different) range element.

That means no two $x$ values map to the same $y$ value.

Note that one to one functions must pass a horizontal line test. That is, a horizontal line cannot intersect the graph more than once. One to one functions are of course functions, so they must pass a vertical line test as well.

State whether the following functions are one to one. Assume each function has a domain of all real numbers unless stated otherwise.

* $f\left(x\right)=x$
* $f\left(x\right)=|x|$
* $f\left(x\right)=x^{2}$
* $f\left(x\right)=x^{2}$ with domain $[0,\infty )$
* $f\left(x\right)=x^{2}$ with domain $(-\infty ,0]$
* $f\left(x\right)=x^{3}$
* $f\left(x\right)=\sqrt{x}$
* $f\left(x\right)=\left⟦x\right⟧$
* $f\left(x\right)=\sqrt[3]{x}$

**Definition:** Let $f$ be a one to one function. Then $f^{-1}$ is the inverse function of $f$ if and only if $\left(f∘f^{-1}\right)\left(x\right)=\left(f^{-1}∘f\right)\left(x\right)=x$ for every $x$ in the domain.

* Note that not all functions have inverses. To have an inverse, the function must be one to one.
* This definition seems complicated, but all it really says is that if a function does something to the number you put in, then $f^{-1}$ will be the function that takes you back to the number you started with. This agrees with our previous knowledge of inverse functions such as multiplication and division, addition and subtraction, square and square root, etc.
* The notation $f^{-1}$ defines a function. It has nothing to do with negative exponents of numbers, which we previously learned were reciprocals. For example, if $f\left(x\right)=x^{2}$ with domain $[0,\infty )$, then $f^{-1}\left(x\right)=\sqrt{x}$. Whereas $3^{-1}=\frac{1}{3}$. You may notice that $3$ and $\frac{1}{3}$ are specific types of inverses (with respect to the multiplication operation).

Examples: Decide whether or not each function is one to one. If so, determine the inverse of the function. State the domain and range of both the function and its inverse, if the inverse exists.

* $f=\{\left(0,2\right),\left(-2,3\right),\left(4,7\right)\}$
* $f\left(x\right)=2x-6$
* $f\left(x\right)=x^{2}-5$ with domain $(-\infty ,\infty )$
* $f\left(x\right)=|x|$
* $f\left(x\right)=x^{2}-5$ with domain $[0,\infty )$
* $f\left(x\right)=\frac{2}{x-3}$

What is the relationship between the domain and range of a function and its inverse? This is important to understand.

Show, from the definition of inverse functions, whether or not the following functions are inverses of each other.

$f\left(x\right)=3x^{3}+5$ and $g\left(x\right)=\frac{1}{3}\sqrt[3]{x-5}$

Sometimes it can be difficult to determine the inverse of a function by inspection. In those cases, use the fact that the domain and range will be switched to find the inverse. That is, switch $x$ and $y$, then solve for $y$. That is then the inverse function. This process always works, so if you prefer to use it instead of our previous line of reasoning, go for it.

Find the inverse of $f\left(x\right)=\frac{2x+3}{5x-7}$

Graph some of the functions and their inverses from the above examples. What is the geometric relationship between a function and its inverse?