3.3 Notes: Zeros of Polynomial Functions

**Factor Theorem** – Let $f(x)$ be a polynomial function. Then, $x-k$ is a factor of $f(x)$ if and only if $f\left(k\right)=0$.

Proof:

**Fundamental Theorem of Algebra** – Every polynomial function of degree 1 or more has at least one complex zero.

The proof of the fundamental theorem uses advanced analysis that is beyond the scope of this course. Carl Gauss, the great German mathematician proved it as part of his P.h.D dissertation.

**Corollary to the Fundamental Theorem** – A polynomial function of degree $n$ has at most $n$ distinct zeros. Further, it has exactly $n$ zeros if multiplicities are counted.

**Conjugates Review**

Recall that if $z=a+bi$, then the conjugate of $z$, denoted $\overbar{z}=a-bi$.

Examples.

* If $z=-3+2i$, then $\overbar{z}=$
* If $z=5-3i$, then $\overbar{z}=$

**Properties of conjugates**

Let $z\_{1}$ and $z\_{2}$ be complex numbers. Then the following are true.

* $\overbar{z\_{1}+z\_{2}}=\overbar{z\_{1}}+\overbar{z\_{2}}$
* $\overbar{z\_{1}z\_{2}}=\overbar{z\_{1}}∙\overbar{z\_{2}}$
* $\overbar{z^{n}}=(\overbar{z})^{n}$

Proof of 2nd statement:

**Conjugate Zeros Theorem** – If $f(x)$ is a polynomial with real coefficients and $z$ is a zero of $f(x)$, then $\overbar{z}$ is also a zero of $f(x)$.

Examples: Decide whether the second polynomial is a factor of the first.

* $f\left(x\right)=-3x^{3}+5x^{2}+11x-18$, $x-2$
* $f\left(x\right)=2x^{3}-17x+10$, $x+3$

Factor $f(x)$ into linear factors given that $k$ is a zero of $f(x)$.

* $f\left(x\right)=3x^{3}-5x^{2}-75x+125$, $k=5$
* $f\left(x\right)=2x^{3}-3x^{2}-18x+27$, $k=-3$
* $f\left(x\right)=x^{4}+3x^{3}-42x^{2}-172x-168$, $k=-2$ (multiplicity 2)

Find all zeros of each polynomial. If necessary, state the multiplicities.

* $f\left(x\right)=x^{3}+3x^{2}-10x-24$ given that $-2$ is a zero.
* $f\left(x\right)=x^{3}-x^{2}-8x+12$ given that $2$ is a zero.
* $f\left(x\right)=x^{3}+7x^{2}+14x+20$ given that $-5$ is a zero.
* $f\left(x\right)=6x^{3}+\left(19-6i\right)x^{2}+\left(16-7i\right)x+(4-2i)$, given that $-2+i$ is a zero of $f(x)$.
* $f\left(x\right)=x^{4}-4x^{3}+6x^{2}-4x+5$ given that $2-i$ is a zero.

Find all complex zeros of $f\left(x\right)=(x-2)^{2}(x+3)^{3}(x^{2}+4)$ and state their multiplicities.

Write a polynomial equation with real coefficients that has only the following as zeros:

$2+3i$ and $2-3i$