1.3: Complex Numbers

Notice that there is no real solution to $x^{2}=-1$ since the square of all real numbers is greater than or equal to zero. Let’s examine this a little closer.

* If $x=0$, then $x^{2}=0$
* If $x>0$, then $x^{2}>0$
* If $x<0$, then $x^{2}>0$
* Thus, for any real number (positive, negative, or zero), none of them have a square that is negative.
* So, if there is a solution to $x^{2}=-1$, then it cannot be a real number.

No problem, we will define a new set of numbers that are not real to use as solutions for equations like this. We define $i=\sqrt{-1}$ with the property that $i^{2}=(\sqrt{-1})^{2}=-1$ so that $i$ is a solution to this equation. $i$ is known as the imaginary unit and is the basis of the set of complex numbers. Notice that $-i$ is also a solution to the equation $x^{2}=-1$.

**Definition** – A complex number is any number that can be written in the form $a+bi$ where $a$ and $b$ are real numbers. Note that all real numbers, $a$, are also complex since they can be written as $a+0i$. Thus, the real numbers are a subset of the complex numbers.

If we recall that $A⊆B$ means that $A$ is a subset of $B$, then we have

$N⊆Z⊆Q⊆R⊆C$,

which shows that every natural number is an integer, every integer is rational, every rational number is real, and every real number is complex.

A complex number that can be written in the form $bi$ is a pure imaginary number.

A nonreal complex number can be written in the form $a+bi$, where $a\ne 0$ and $b\ne 0$

**Important Note:** Let $a$ be a positive real number. Then, $\sqrt{-a}=\sqrt{-1}\sqrt{a}=i\sqrt{a}$. While $\sqrt{a}i$ is an equivalent expression, it is not commonly used because it could be easily confused with $\sqrt{ai}$, which is different.

Example: $\sqrt{-4}\sqrt{-9}=i\sqrt{4}∙i\sqrt{9}=i^{2}∙2∙3=-6$.

Note in this example it would have been incorrect to use the negative times negative equals positive convention that is familiar for real numbers to get $6$. Remember that $\sqrt{-4}$ and $\sqrt{-9}$ are not real numbers, so they are not “negative” and that property is not valid. The complex numbers have many of the same properties as real numbers such as commutativity, associativity, identity, and inverses of addition and multiplication as well as a distributive property. What’s missing is an order. Reals can be ordered on a number line, the complex numbers can’t.

Examples: Simplify each of the following and write all complex numbers with $i$ notation.

* $\sqrt{-49}$
* $-\sqrt{-49}$
* $\sqrt{49}$
* $-\sqrt{49}$
* $\sqrt{-20}$
* $\sqrt{-10}∙\sqrt{-10}$
* $\frac{\sqrt{-150}}{\sqrt{-2}}$
* $\frac{\sqrt{-90}}{\sqrt{20}}$
* $\frac{\sqrt{-18}}{\sqrt{-32}}$
* $\frac{\sqrt{-21}∙\sqrt{-3}}{\sqrt{7}}$
* $\frac{6+\sqrt{-18}}{3}$
* $\frac{6-\sqrt{-24}}{12}$

Addition of complex numbers works just as you’d imagine. $\left(a+bi\right)+\left(c+di\right)=\left(a+c\right)+\left(b+d\right)i$. Just combine like terms.

Examples of addition/subtraction:

* $\left(2-3i\right)+\left(-5+2i\right)$
* $\left(2-3i\right)-(-5+2i)$
* $\left(1+6i\right)+\left(-2-i\right)-(4-2i)$
* $4\sqrt{-20}-3\sqrt{-5}+\sqrt{-45}$

Multiplication of complex numbers can be done by using the distributive property and $i^{2}=-1$.

Examples of multiplication:

* $i(2+3i)$
* $(5+3i)(4-2i)$
* $(-2+i)(i+3)$
* $(-4+3i)^{2}$
* $i(2-3i)(4+2i)$
* $(3-i)(-3+3i)(4i)$
* $(4-3i)(1+2i)(1-i)$

Division is trickier because the denominator is usually a radical. Multiply the numerator and denominator by the conjugate of the denominator to eliminate the radical denominator 1st. Let’s define what is meant by a conjugate before going further. If $a+bi$ is a complex number, then it’s conjugate is $a-bi$, the same first part but opposite second part.

Examples of conjugates. Fill in the table.

|  |  |
| --- | --- |
| Complex Number | Conjugate |
| $$3-5i$$ |  |
| $$2+2i$$ |  |
| $$-8-15i$$ |  |
|  | $$-3+4i$$ |
| $$4-i\sqrt{2}$$ |  |
| $$i$$ |  |

Examples of division: $\frac{(2+3i)}{(1-4i)}=\frac{(2+3i)(1+4i)}{(1-4i)(1+4i)}=\frac{(-10+11i)}{17}=-\frac{10}{17}+\frac{11}{17}i$

* $\frac{3}{-i}$
* $\frac{8-3i}{i}$
* $\frac{-2-3i}{-i}$
* $\frac{i}{2-3i}$
* $\frac{4+5i}{2-3i}$
* $\frac{-9+5i}{6-8i}$

Powers of $i$ are interesting to look at because a neat pattern emerges. See if you can notice it.

|  |  |
| --- | --- |
| Powers of $i$ | Simplified Form |
| $$i$$ |  |
| $$i^{2}$$ |  |
| $$i^{3}$$ |  |
| $$i^{4}$$ |  |
| $$i^{5}$$ |  |
| $$i^{6}$$ |  |
| $$i^{7}$$ |  |
| $$i^{8}$$ |  |
| $$i^{9}$$ |  |
| $$i^{10}$$ |  |
| $$i^{11}$$ |  |
| $$i^{12}$$ |  |

Examples of Powers of $i$

* $i^{23}$
* $i^{84}$
* $i^{205}$
* $i^{-3}$
* $\frac{1}{i^{-26}}$

Is a number a solution to an equation? Do you remember when we asked whether $\{3\}$ was a solution to $2x-6=0$? Do you remember how to answer this?

Let’s now ask a similar question. Is $-2+2i$ a solution to $x^{2}+4x+8=0$?

In the first equation, $2x-6=0$, you should remember a method to solve this so you can find the solution. There’s also a method that we can use to solve the second equation, $x^{2}+4x+8=0$. You may remember and you may not. We’ll discuss this in class soon, but for now the best we can do is check that a given number is a solution.